A Ranking Method for Fuzzy Complementary Judgment Matrix

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Abstract

Fuzzy Analytic Hierarch Process (FAHP) was developed to solve imprecise hierarchical problems. The lack of consistency in decision making can lead to inconsistent conclusions. It is difficult to ensure a consistent pair-wise comparison. This paper proposes an analysis for setting up a goal programming model to obtain the ranking vector of alternative and the consistent fuzzy judgment matrix for approximating to the decision maker’s preference information.

Keywords: Fuzzy judgment matrix; consistency; goal programming.

Introduction

Multi-attribute decision-making (MADM) addresses the problem of choosing an optimum choice that has been highest degree of satisfaction from a set of alternatives that are characterized in terms of their attributes. FAHP extension of Analytic Hierarch Process (AHP) was developed to solve imprecise hierarchical problems. As part of AHP procedure, a consistency check is required to identify inconsistency matrix. But, the lack of consistency in decision making can lead to inconsistent conclusions. Therefore, the purpose of this research is to establish a linear goal programming model and analysis for ordinal consistency of fuzzy complementary judgment matrix problem.

Literature review

Lamata and Pelaez [1] defined the Consistency Index (CI) of a matrix using the average of the consistency index of the matrix triplets. Li and Ma [2] developed a model that can assist on marking a consistent decision and used Gower plots to judge the ordinal consistency graphically. Basile and Dapuzzol [3] used the complete strict simple order to judge the ordinal consistency of judgment matrix. Luo [4] studied the revising method of judgment matrix and considered that ordinal consistency was the prerequisite of ordinal consistency. Preference relations are the most common representation of information used for solving decision making problems due to their effectiveness in modeling processes. These preference relations can be categorized into multiplicative preference relations ([5], [6]), Fuzzy preference relations ([5], [7] ~ [10]) and linguistic preference relations ([11] ~ [15]). Zhu et al. [16] demonstrated that consistent analysis should be based on ordinal consistency. On the basis of AHP consistent judgment matrix, Wang and Guo [17] directly determine the ideal priority vector by a general formula for solving the fuzzy judgment matrix priority. Zhang et al. [18] proposed a method through the no-transitive rout number and no-transitive route contribution number for solving the fuzzy judgment matrix without ordinal consistency. Wu [19] used utilizing the continuous interval argument operator and the expected function for ranking, which is able to be adjusted according to decision-maker’s optimistic degree and will be more reasonable. Fan and Jiang [20] in an overview on ranking method of fuzzy judgment matrix proposed a new concept of consistency based on the new consistency approximation and ranking method to avoid misleading conclusions. This characterization simplifies the analysis of consistency among expert opinions. Thus, this study applies linear goal programming method to enhance the consistency of the fuzzy AHP method. The proposed method...
yields decision matrices for making pair-wise comparisons using additive reciprocal property and consistency.

**Preliminary Knowledge**

**Definition 1:** Suppose $A = (a_{ij})_{n \times n}$ be a judgment matrix given by a decision-maker. If $0 \leq a_{ij} \leq 1$ for $i, j = 1, 2, ..., n$, then $A = (a_{ij})_{n \times n}$ is called a fuzzy judgment matrix, where $a_{ij}$ is the preference degree of $x_i$ to $x_j$.

$$a_{ij} = \begin{cases} 0 & x_i \text{ is strictly inferior to } x_j \\ (0, 0.5) & x_i \text{ is inferior to } x_j \\ 0.5 & x_i \text{ is identical to } x_j \\ (0.5, 1) & x_i \text{ is superior to } x_j \\ 1 & x_i \text{ is strictly superior to } x_j \end{cases} \quad (1)$$

Fuzzy judgment matrix shows the preference of periwigs comparison on alternatives. If a fuzzy judgment matrix is not ordinal consistent, the fuzzy judgment matrix is not acceptable [21].

**Definition 2:** Suppose $A = (a_{ij})_{n \times n}$ be a fuzzy judgment matrix. $A = (a_{ij})_{n \times n}$ is called fuzzy complementary judgment matrix if it follows.

$$a_{ij} + a_{ji} = 1, \forall i, j = 1, 2, ..., n \quad (2)$$

$$a_{ii} = 0.5$$

**Definition 3:** Suppose $A = (a_{ij})_{n \times n}$ be a fuzzy complementary matrix. If the following condition is true, $A = (a_{ij})_{n \times n}$ is ordinal consistent.

$$a_{ik} > 0.5, a_{kj} > 0.5 \Rightarrow a_{ij} > 0.5 \quad (3)$$

or

$$a_{ik} < 0.5, a_{kj} < 0.5 \Rightarrow a_{ij} < 0.5$$

**Definition 4:** Suppose $A = (a_{ij})_{n \times n}$ be a fuzzy complementary fuzzy matrix. If the following condition is true, for $i, j, k = 1, 2, ..., n$, then $A = (a_{ij})_{n \times n}$ is consistency of fuzzy complementary judgment matrix.

$$a_{ij} = a_{ik} - a_{jk} + 0.5 \quad (4)$$

**Proposition 1:** Suppose $A = (a_{ij})_{n \times n}$ be a fuzzy complementary fuzzy matrix. The ranking weight vector is $W = (w_1, w_2, ..., w_n)^T$ and
\[
a_{ij} = \frac{W_i}{W_i + W_j}
\]

(5)

Proof: Use (4), we prove \(a_j + a_{jk} - a_{ik} = 0.5\)

\[
a_j + a_{jk} - a_{ik} = \frac{W_i}{W_i + W_j} + \frac{W_j}{W_j + W_k} - \frac{W_i}{W_i + W_k} = 0.5
\]

\[
= \frac{w_i(w_j + w_k)(w_j + w_k) + w_j(w_i + w_j)(w_i + w_k)}{(w_i + w_j)(w_i + w_k)} - \frac{w_i(w_j + w_k)(w_j + w_k)}{(w_i + w_j)(w_i + w_k)}
\]

\[
= 0.5
\]

**Linear Goal Programming Mode** [22]

Suppose alternative \(x_i\) have ranking value \(w_i \quad \forall i = 1,2,\ldots,n\), and \(w_i \geq 0, \sum_{i=1}^{n} w_i = 1\). the matrix \(A = (a_{ij})_{n \times n}\) is consistency of fuzzy complementary judgment matrix. Set

\[
a_j = \frac{w_j}{w_i + w_j}
\]

(6)

Using (6), we have

\[
(1-a_j)w_j = a_j w_j
\]

(7)

We use \(\frac{w_j}{w_i + w_j}\) approximation to \(a_j\) for calculation \(w_i \quad \forall i = 1,2,\ldots,n\). That is

\[
a_j \approx \frac{w_j}{w_i + w_j} \quad \forall i, j, k = 1,2,\ldots,n
\]

(8)

From (8), we know that there have same characters between \(a_j\) and \(\frac{w_j}{w_i + w_j}\), and build the following objective model.

\[
\min Z_j = \left| (1-a_j)w_j - a_j w_j \right|, \quad \forall i, j, k = 1,2,\ldots,n; \quad i \neq j
\]

(9a)

s.t. \(\sum_{j=1}^{n} w_j = 1\)

(9b)

\(w_j \geq 0, \quad \forall i = 1,2,\ldots,n\)

(9c)

In the above model, we can write \(Z_j\) and the expected value of \(z_i\) approach to 0. \(\forall i, j = 1,2,\ldots,n; \quad i \neq j\).

We transform the (9a)–(9c) to the following linear goal programming problem.

\[
\min z = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} s_j d_{ij} + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} t_j d_{ij}
\]

(10a)

s.t. \((1-a_j)w_j - a_j w_j - d_{ij}^+ + d_{ij}^- = 0 \quad \forall i, j = 1,2,\ldots,n; \quad i \neq j\)

(10b)

\(\sum_{j=1}^{n} w_j = 1\)

(10c)

\(w_j \geq 0, \quad \forall i = 1,2,\ldots,n\)

(10d)
\[ d_{ij}^+ \geq 0, \quad d_{ij}^- \geq 0, \quad \forall i, j = 1, 2, \ldots, n; \quad i \neq j \quad (10e) \]

Where \( d_{ij}^+ \) is deviation of variable that the objective function \( z_j \) is lower than expected value 0. \( d_{ij}^- \) is deviation of variable that the objective function \( z_j \) is higher than expected value 0. \( s_{ij} \) and \( t_{ij} \) are weight coefficient of \( d_{ij}^+ \) and \( d_{ij}^- \) respectively.

We see that (10b) includes the following constraint conditions:

\[(1 - a_{ij})w_i - a_{ij}w_j - d_{ij}^+ + d_{ij}^- = 0 \quad \forall i, j = 1, 2, \ldots, n; \quad i < j \quad (11a)\]

\[(1 - a_{ij})w_i - a_{ij}w_j - d_{ij}^+ + d_{ij}^- = 0 \quad \forall i, j = 1, 2, \ldots, n; \quad i > j \quad (11b)\]

According to \( a_{ij} + a_{ji} = 1 \), we can prove (11a) equivalence to (11b). Therefore, we actual consider the \( n(n-1)/2 \) constraint conditions. In addition, we can let Eq. (10a) all of the objective function is fair competition and there is no preference relation. Therefore, we take \( s_{ij} = t_{ij} = 1, \forall i, j = 1, 2, \ldots, n; \quad i < j \).

The above model can write as:

\[
\min z = \sum_{i=1}^{n} \sum_{j<i}^{n} (d_{ij}^+ + d_{ij}^-) \quad (12a)
\]

\[
(1 - a_{ij})w_i - a_{ij}w_j - d_{ij}^+ + d_{ij}^- = 0 \quad \forall i, j = 1, 2, \ldots, n; \quad i < j \quad (12b)
\]

\[
\sum_{j=1}^{n} w_j = 1 \quad (12c)
\]

\[
w_j \geq 0 \quad \forall i = 1, 2, \ldots, n \quad (12d)
\]

\[
d_{ij}^+ \geq 0, \quad d_{ij}^- \geq 0, \quad \forall i, j = 1, 2, \ldots, n; \quad i \neq j \quad (12e)
\]

Based on the fuzzy judgment matrix \( A = (a_{ij})_{n \times n} \), it established the linear goal programming model. By solving this model, we can get the required ordering vector \( w = (w_1, w_2, \ldots, w_n)^T \). We obtain the ranking vector of alternatives and the consistent fuzzy judgment matrix for approximating to the decision maker’s preference information.

**Illustration**

Suppose expert gives his preference on alternatives \( X = \{x_1, x_2, x_3, x_4\} \). The fuzzy complementary matrix from Chiclana, et al. [23] is:

\[
A = \begin{bmatrix}
0.5 & 0.1 & 0.6 & 0.7 \\
0.9 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.9 \\
0.3 & 0.6 & 0.1 & 0.5
\end{bmatrix} \quad (13)
\]

Using (12a) ~ (12e), the linear goal programming model can be obtained and is shown in (14) as follows:
\[
\begin{align*}
\min & \quad Z = d_{12}^+ + d_{13}^+ + d_{14}^+ + d_{23}^+ + d_{24}^+ + d_{34}^+ + d_{12}^- + d_{13}^- + d_{14}^- + d_{23}^- + d_{24}^- + d_{34}^- \\
\text{s.t.} & \quad 0.9w_1 - 0.1w_2 - d_{12}^+ + d_{12}^- = 0 \\
& \quad 0.4w_1 - 0.6w_3 - d_{13}^+ + d_{13}^- = 0 \\
& \quad 0.3w_1 - 0.7w_4 - d_{14}^+ + d_{14}^- = 0 \\
& \quad 0.2w_2 - 0.8w_3 - d_{23}^+ + d_{23}^- = 0 \\
& \quad 0.6w_2 - 0.4w_4 - d_{24}^+ + d_{24}^- = 0 \\
& \quad 0.1w_3 - 0.9w_4 - d_{34}^+ + d_{34}^- = 0 \\
& \quad w_j \geq 0 \quad \forall i = 1, 2, ..., 4 \\
& \quad d_{12}^+, d_{13}^+, d_{14}^+, d_{23}^+, d_{24}^+, d_{34}^+ \geq 0 \\
& \quad d_{12}^-, d_{13}^-, d_{14}^-, d_{23}^-, d_{24}^-, d_{34}^- \geq 0
\end{align*}
\]

Using linear programming method (MatLab software), we obtained the solution of model are:

\[
\begin{align*}
& w_1 = 0.0789, w_2 = 0.7098, w_3 = 0.1775, w_4 = 0.0338, \\
& d_{12}^+ = 0.0, \quad d_{13}^+ = 0.0749, \quad d_{14}^- = 0.0, \\
& d_{23}^- = 0.0, \quad d_{24}^- = 0.01268 \quad (14) \\
& d_{12}^- = 0.0, \quad d_{13}^+ = 0.0, \quad d_{14}^- = 0.0, \\
& d_{23}^+ = 0.0, \quad d_{24}^+ = 0.8412, \quad d_{34}^- = 0.0.
\end{align*}
\]

We get the required ordering vector \( w = (0.0789, 0.7098, 0.1775, 0.0338)^T \).

Using (8), we can construct a consistency of fuzzy complementary judgment matrix.

\[
R = \begin{bmatrix} 0.5 & 0.1 & 0.307 & 0.7 \\ 0.9 & 0.5 & 0.719 & 0.955 \\ 0.693 & 0.281 & 0.5 & 0.84 \\ 0.3 & 0.045 & 0.416 & 0.5 \end{bmatrix}
\]

The ranking vector of alternatives is \( x_2 \succ x_3 \succ x_1 \succ x_4 \).

**Conclusion**

The fuzzy consistent judgment matrix is analyzed in this paper, which are linear goal programming mode and presented a formula for priority of fuzzy complementary judgment matrix. The conclusion of this paper is helpful to the correct application of the making formula of fuzzy consistent matrix, enriching the theory and method of fuzzy decision making.
References


